# Analysis SEP 2026: Measure Theory (Part I)

## 1 Warm up

**Problem 1.06.04** Let $f \in L^1(\mathbb{R})$. Prove that

$$\lim_{\lambda \in \mathbb{R},\, \lambda \to \infty} \int_{\mathbb{R}} f(x) e^{-i\lambda x} = 0.$$

## 2 Exercises

**Problem 1.25.6** Let $\mu$ be a (positive) measure on $\mathbb{R}$ such that $\mu(\mathbb{R}) < \infty$. Define

$$G(x) = \mu((-\infty, x])$$

and let $\beta \in \mathbb{R}$. Evaluate

$$\int_{-\infty}^{\infty} (G(x+\beta) - G(x))\, dx.$$

**Problem 8.07.5** (i) Prove the identity

$$\{y : y \in E_k \text{ for infinitely many } k\} = \bigcap_{n=1}^{\infty} \bigcup_{k \geq n} E_k.$$

(ii) Let $A$ be the set of points $x \in (0,1)$ with the property that there are infinitely many fractions $p/q$ with integers $p, q$ such that

$$|x - p/q| < 1/q^3.$$

Show that $A$ is a set of measure 0.

**Problem 1.25.4** Let $f$ be a measurable function on $[0,1]$ such that $f(x) > 0$ for all $x \in [0,1]$ and assume that $\int_{[0,1]} f(x)dx < \infty$. Let $0 < \gamma < 1$. Prove

$$\inf_{|E|=\gamma} \int_E f(x)\, dx > 0.$$

Here the infimum is taken over all measurable subsets of $[0,1]$ of Lebesgue measure $\gamma$.

**Problem 8.16.6** Let $f$ be a non-negative measurable function on $[0, 1]$. Prove that the following statements are equivalent.

(1) There exists $a > 0$ such that

$$\int_0^1 e^{af(x)}\, dx < \infty.$$

(2) There exists $C > 0$ such that

$$\left(\int_0^1 f(x)^p\, dx\right)^{1/p} \leq Cp \quad \text{for all} \quad 1 \leq p < \infty.$$

**Problem 1.17.5** Let $P(x, y) = \sum_{m=0}^{M} \sum_{n=0}^{N} a_{mn} x^m y^n$ be a real valued polynomial on $\mathbb{R}^2$ (which is not identically zero). Prove that the set

$$\{(x, y) \in \mathbb{R}^2 : P(x, y) = 0\}$$

has Lebesgue measure zero.